Second-order Quantile Methods for Online Sequential Prediction

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Outline

- Prediction with Expert Advice
  - Setting
  - Standard algorithm (and its limitations)
- Improvements
  - Either second-order bounds
  - Or quantile bounds
- How to get both improvements
- Online shortest path
Sequential Prediction with Expert Advice

- $K$ experts *sequentially* predict data $x_1, x_2, \ldots$
- Goal: predict (almost) as well as the best expert on average
Sequential Prediction with Expert Advice

- $K$ experts **sequentially** predict data $x_1, x_2, \ldots$
- Goal: predict (almost) as well as the **best expert** on average
- Applications:
  - online **classification**, e.g. spam detection
  - online convex **optimization**
  - **boosting**
  - **differential privacy**
  - predicting time series like electricity consumption or air pollution levels
Formal Setting

- Every round $t = 1, \ldots, T$ :
  1. Predict **probability distribution** $w_t$ on $K$ experts
  2. Observe expert losses $\ell_t^1, \ldots, \ell_t^K \in [0, 1]$
  3. Our expected loss is $\hat{\ell}_t = \mathbb{E}_{w_t(k)}[\ell^k_t]$
Formal Setting

- Every round $t = 1, \ldots, T$:
  1. Predict probability distribution $w_t$ on $K$ experts
  2. Observe expert losses $\ell^1_t, \ldots, \ell^K_t \in [0, 1]$
  3. Our expected loss is $\hat{\ell}_t = \mathbb{E}_{w_t(k)}[\ell^k_t]$

- Goal: small regret for every expert $k$

$$R^k_T = \sum_{t=1}^{T} \hat{\ell}_t - \sum_{t=1}^{T} \ell^k_t$$
Standard Algorithm

- **Exponential weights** with **prior** $\pi$:

  $$w_t(k) = \frac{\pi(k)e^{\eta R_{t-1}^k}}{\text{normalisation}}$$

- **learning rate** $\eta$ is a parameter
  - large $\eta$: aggressive learning
  - small $\eta$: conservative learning
Standard Algorithm

- **Exponential weights with prior** \( \pi \):
  \[
  w_t(k) = \frac{\pi(k)e^{\eta R_{t-1}^k}}{\text{normalisation}}
  \]

- **Learning rate** \( \eta \) is a parameter
  - large \( \eta \): aggressive learning
  - small \( \eta \): conservative learning

- \( w_t \) is gradient of potential function
  \[
  \ln \sum_k \pi(k)e^{\eta R_{t-1}^k}
  \]
Basic Regret Guarantee

- For learning rate $\eta = \sqrt{8 \ln(K)/T}$

$$R^k_T < \sqrt{T \ln(K)}$$ for all experts $k$

- Average regret per round goes to 0
- $T$ does not measure inherent difficulty
- $\ln(K)$ does not count effective nr of experts
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Improvement 1: Second-order Bounds

\[ R_T^k < \sqrt{T \ln(K)} \]

- \( T \) simply counts nr of rounds
- Want to replace by some measure of the variance in the losses
- Different proposals [Cesa-Bianchi, Mansour, Stoltz, 2007], [Hazan, Kale, 2010], [Gaillard, Stoltz, vE, 2014]

\[ V_T^k := \sum_{t=1}^{T} (\hat{\ell}_t - \ell_t^k)^2 \]
Improvement 1: Second-order Bounds

\[ R_T^k \prec \sqrt{V_T^k \ln(K)} \]

- Different specialized algorithms
- Different clever tricks to choose learning rate adaptively over time

[Cesa-Bianchi, Mansour, Stoltz, 2007], [Hazan, Kale, 2010], [Gaillard, Stoltz, vE, 2014]
Improvement 2: Quantile Bounds

\[ R_T^k \lesssim \sqrt{T \ln(K)} \]

- \( \ln(K) \) is nr. of bits to identify best expert
- But suppose multiple experts \( \mathcal{K} \subset \{1, \ldots, K\} \) are all good
- Want to replace by

\[ \ln \frac{1}{\pi(\mathcal{K})} \]

for prior \( \pi \) on experts

[Chaudhuri, Freund, Hsu, 2009]
Improvement 2: Quantile Bounds

\[ R_T^K \leq \sqrt{T \ln \frac{1}{\pi(\mathcal{K})}} \]

- \( R_T^K = \min_{k \in \mathcal{K}} R_T^k \) good when all \( k \in \mathcal{K} \) good
- Specialized algorithms
- Clever tricks to choose learning rate adaptively over time

[Chaudhuri, Freund, Hsu, 2009]
Both Improvements: Second-order Quantile Bounds?

\[ R_T^\mathcal{K} \lesssim \sqrt{V_T^\mathcal{K} \ln \frac{1}{\pi(\mathcal{K})}} \]

- Different specialized algorithms
- Incompatible clever tricks to tune learning rate

Need something simple!
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New Algorithm

• Exponential weights with prior $\pi$:

$$w_t(k) = \frac{\pi(k)e^{\eta R_t^k}}{\text{normalisation}}$$

• 1. Incorporate variance:

$$w_t(k) = \frac{\pi(k)e^{\eta R_t^k - \eta^2 V_t^k}}{\text{normalisation}}$$
New Algorithm

- **Exponential weights with prior** $\pi$:

$$ w_t(k) = \frac{\pi(k) e^{\eta R_t^k}}{\text{normalisation}} $$

- **2. Add prior on learning rates:**

$$ w_t(k) = \frac{\int \gamma(\eta) \pi(k) e^{\eta R_t^k - \eta^2 V_t^k} \eta \, d\eta}{\text{normalisation}} $$
New Regret Bound

Thm. Any \( \pi(k) \), right choice of \( \gamma(\eta) \) achieves

\[
R_T^\mathcal{K} < \sqrt{V_T^\mathcal{K} \left( \ln \frac{1}{\pi(\mathcal{K})} + \ln \ln T \right)}
\]

for all \( \mathcal{K} \)

- Averages under the prior instead of worst in \( \mathcal{K} \):

\[
R_T^\mathcal{K} = \mathbb{E}_{\pi(k|\mathcal{K})} [R_T^k] \quad V_T^\mathcal{K} = \mathbb{E}_{\pi(k|\mathcal{K})} [V_T^k]
\]
New Regret Bound

Thm. Any $\pi(k)$, right choice of $\gamma(\eta)$ achieves

$$R_T^K \prec \sqrt{V_T^K} \left( \ln \frac{1}{\pi(K)} + \ln \ln T \right)$$

for all $K$

- **Averages** under the prior instead of worst in $K$:

  $$R_T^K = \mathbb{E}_{\pi(k|K)}[R_T^k] \quad V_T^K = \mathbb{E}_{\pi(k|K)}[V_T^k]$$

- If $T = \text{age of universe in \mu s}$: $\ln \ln T < 4$
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Learn Shortest Path in a Graph

- Every round $t$, each edge incurs loss
- $\nu$: best distribution on paths
- Goal: learn $\nu$

$$R_T^\nu < \sqrt{V_T^\nu} \left( \text{complexity}(\nu) + K \ln \ln \ln T \right)$$
Summary

- Improvements over standard exponential weights algorithm:
  - Either second-order bounds
  - Or quantile bounds
- New algorithm gets both improvements
  - Surprisingly simple generalization of exponential weights
- Extension to online shortest path
  (for other combinatorial problems, come to poster)